Malaysian Journal of Mathematical Sciences 18(4): 949–956 (2024)
https://doi.org/10.47836/mjms.18.4.15

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

On the Relation Between Topological Free and Topological Dual Injective Modules

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Received: 4 April 2024
Accepted: 8 August 2024

Abstract

Algebraic topology is a branch of mathematics which use the concepts of abstract algebra to study topological spaces in which to find algebraic invariants that classify topological spaces up to homeomorphism. In this paper, the basic properties of some concepts on algebraic topology such as the topological ring, the topological module, and the topological free module were recalled, which were helped to define new concepts and proof some of there properties as a new results. Then, some results related to the relation between the free topological module and homomorphism topology. As in any application will be introduced, the tensor concept will be chosen to explain the new results related to the topological module.

Keywords: topological ring; topological module; topological free module; homomorphism topological module; topological dual injunctive.

1 Introduction

Topology and algebra are the two fundamental branches of mathematics that play complementary roles; topology deals with the concepts of convergence, continuity, and limit. While, the algebra studies all the operations and provides a basis for algorithms and calculations [3]. One of such instance of using a special rings in applications is the boolean ring RB for which $a^2 = a$ for all $a \in RB$. The Boolean ring plays a vital role in the areas of engineering communication and computer science [8]. Topology and algebra come into contact most naturally in applications in higher-level domains of mathematics. Some examples of algebraic topology include Homology theory. This is a fundamental tool in algebraic topology that associates a sequence of abelian groups with a topological space. The definition of a topological group was first defined by Kaplanski in 1955 [4]. Then, Ibrahim and Khalaf [5] introduced the definitions of a topological module and a topological submodule.

After that, these concepts were studied by many scientists such as Alberto Tolono and Nelson. Mahuoub [6] introduced the definition of the concept of a topological injective module and gave some of its properties [7]. In this paper, we begin with recalling some background information on topological group, topologica ring and topological module [5], that will be used in this work, as well as reviewing the definition of topological module base and topological free module[1] which we extend it by choosing tensor concept of topological free modules. Some new results have been proved the equevalency of the conditions of a tensor product of two topologically free modules. As well as, we show that the homomorphic image of n tensor product of $FR-$module is $FR-$module. Another important result has been proved is the n tensor product of $FR-$module is completely disconnected and hence it is totally disconnected.

2 Background

In this section, we introduce some basic concepts of the most important definitions that will be used in this research.

Definition 2.1. [1, 7]. *An abelian group $\mathcal{G}$ with a topology τ on $\mathcal{G}$ is called a topological group, if it satisfies the two conditions that, if for each two elements a and b in $\mathcal{G}$ such that $a + b \in \mathfrak{U} \in \tau$ there exist $\mathfrak{V}$ and $\mathfrak{W}$ in τ with $a \in \mathfrak{V}$ and $b \in \mathfrak{W}$ and $\mathfrak{V} + \mathfrak{W} \in \tau$. In addition to the second condition that for each element a of $\mathcal{G}$, $a \in \mathfrak{V} \in \tau$ if its inverse $a^{-1} \in \mathfrak{U} \in \tau$ implies that $\mathfrak{V}^{-1} \subset \mathfrak{U}$.*

Definition 2.2. [2] *A ring $\mathcal{R}$ with a topology τ is called a topological ring if $\mathcal{R}$ is a topological group with respect to addition, and for each two elements a and b of $\mathcal{R}$ and for any $\mathfrak{U} \in \tau$ if $a.b \in \mathfrak{U}$ there exist $\mathfrak{V}, \mathfrak{W} \in \tau$ with $a \in \mathfrak{V}, b \in \mathfrak{W}$ and $\mathfrak{V}, \mathfrak{W} \subset \mathfrak{U}$.*

Definition 2.3. [6, 7] *A left module M on a ring $\mathcal{R}$ is called left topological module on $\mathcal{R}$, if it is a topological group and satisfies the following: for every $x \in \mathrm{M}$, $\lambda \in \mathcal{R}$ the map $h : \mathcal{R} \times \mathrm{M} \to \mathrm{M}$ is continuous.*

Definition 2.4. [1, 7, 2]. *A homomorphism module $f : E \to \acute{E}$, where both E and $\acute{E}$ are topological modules which are called topological module homomorphism, if they are continuous.*

Definition 2.5. [2] *A topological module base $\acute{E}$ of topological module E should be the family $\{E_n\}_{n \in N}$ of the open topological period submodule of E that satisfies the conditions that:*

1. *$\{E_n\}_{n \in N}$ represented essential system of neighborhood for zero in E.*
2. *$\bigcap_{n \in N} \{E_n\} = \{0\}$.*

3. $E_1 \supset E_2 \supset \ldots$
4. $\bigoplus_{v \neq n} \{E_n\} = E$, *if*

$$u_v : E \to E_v,$$

be open continuous projective then:

$$ker\,(u_v) = \bigoplus_{v \neq n} \{E_n\},$$

and

$$E = ker\,(u_v) \bigoplus \{E_n\},$$

where $n \geq v \geq 1$.

Definition 2.6. [1, 6] *A topological module* $\mathfrak{E}$ *of a topological ring* $\mathfrak{R}$ *is called topological free module if it has topological module base.*

Definition 2.7. [9, 7] *Topological modules base* $\acute{B}_1 \otimes \acute{B}_2$ *of topological module* $\acute{E} \otimes \acute{\acute{E}}$ *be the family* $\left\{\acute{E}_n \otimes \acute{\acute{E}}_n\right\}_{n \in N}$ *of open topological period of* $\acute{E} \otimes E_2$ *satisfies the following conditions,*

1. $\left\{\acute{E}_n \otimes \acute{\acute{E}}_n\right\}_{n \in N}$ *is essential of neighborhood of zero in* $\acute{E} \otimes \acute{\acute{E}}$.
2. $\bigcap_{n \in N} \left\{\acute{E}_n \otimes \acute{\acute{E}}_n\right\} = \{0\}$.
3. $\acute{E}_1 \otimes \acute{\acute{E}}_1 \supset \acute{E}_2 \otimes \acute{\acute{E}}_2 \supset \ldots$
4. $\otimes \left(\acute{E}_n \otimes \acute{\acute{E}}_n\right) = \acute{E} \otimes \acute{\acute{E}}$, *if* $u_v : \acute{E} \otimes \acute{\acute{E}} \to \acute{E}_v \otimes \acute{\acute{E}}_v$, *open continuous dual injective map implies that,*

$$ker\,(u_v) = \otimes \left(\acute{E}_n \otimes \acute{\acute{E}}_n\right),$$

and

$$\left(\acute{E} \otimes \acute{\acute{E}}\right) = ker\,(u_v) = \otimes \left(\acute{E}_n \otimes \acute{\acute{E}}_n\right),$$

where $n \geq v \geq 1$.

3 Topological Free and Topological Dual Injective Modules

Definition 3.1. *A topological module* $\left(\acute{E} \otimes \acute{\acute{E}}\right)$ *of a topological ring is called topological free module denoted by* $FR-$ *modules if it has bases of topological modules.*

Remark 3.1. *Every module* $\acute{E} \otimes \acute{\acute{E}}$ *is topological* FR*- module but the opposite is not true.*

Proposition 3.1. *If* $\acute{E} \otimes \acute{\acute{E}}$ *is topological module of topological ring and* $\acute{E} \otimes \acute{\acute{E}}$ *is topological* FR*–module then every surjective homomorphism topological module* $\acute{g} : \acute{A} \otimes \acute{\acute{A}} \to \acute{E} \otimes \acute{\acute{E}}$ *is split for all topological module* $\acute{A} \otimes \acute{\acute{A}}$ *of topological ring.*

Proof. The proof is clear $\square$

Theorem 3.1. *If $\acute{E} \otimes \acute{\acute{E}}$ is topological FR- module of topological ring then the conditions:*

i. $\acute{E} \otimes \acute{\acute{E}}$ has non-empty topological base.

ii. $\acute{E} \otimes \acute{\acute{E}}$ topological tenser product of topological priod submodule and every one of them is homomorphism topological module.

Proof. There exists non-empty family $\acute{B}_1 \otimes \acute{B}_2$ and $i : \acute{B}_1 \otimes \acute{B}_2 \to \acute{E} \otimes \acute{\acute{E}}$ with property if topological module $\acute{A} \otimes \acute{\acute{A}}$ of topological ring and map $\acute{f} : \acute{B}_1 \otimes \acute{B}_2 \to \acute{A} \otimes \acute{\acute{A}}$, $\exists$ uniqe homomorphism topological module $\acute{f} : \acute{E} \otimes \acute{\acute{E}} \to \acute{A} \otimes \acute{\acute{A}}$ such that, $i \circ \acute{\acute{f}} = \acute{f}$ that is mean there exists $\acute{\acute{f}}$ to give commutative as in the Figure 1.

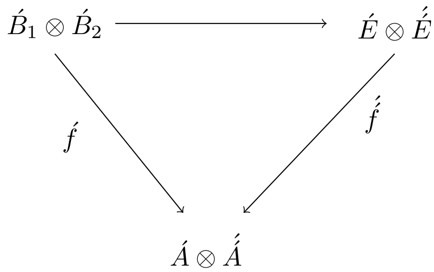

Figure 1: $i \circ \acute{\acute{f}} = \acute{f}$.

$\square$

Proposition 3.2. *If $\acute{E} \otimes \acute{\acute{E}}$, $E_1 \otimes \acute{E}_1$ and $E_2 \otimes \acute{E}_2$ are left topological modules such that $E_1 \otimes \acute{E}_1$ is topological FR- module and $i : E_1 \otimes \acute{E}_1 \to E_2 \otimes \acute{E}_2$ be homomorphism topological module then $E_2 \otimes \acute{E}_2$ is topological FR- module.*

Proof. The proof of this proposition is the same as the proof of Theorem 3.1 through substituted $\acute{A} \otimes \acute{\acute{A}}$ by $\acute{E} \otimes \acute{\acute{E}}$. $\square$

Proposition 3.3. *If $E_1 \otimes \acute{E}_1$, $E_2 \otimes \acute{E}_2$ and $E_3 \otimes \acute{E}_3$ are left topological modules of a ring R such that $E_1 \otimes \acute{E}_1$, $E_2 \otimes \acute{E}_2$ are topological FR–modules and $h : E_1 \otimes \acute{E}_1 \to E_3 \otimes \acute{E}_3$ are homomorphism topological module then $E_3 \otimes \acute{E}_3$ is topological FR–modules.*

Proof. By the diagram we obtain $A_1 \otimes \acute{A}_1$ and $B_1 \otimes \acute{B}_1$ are topological module and,
$\acute{f} : E_1 \otimes \acute{E}_1 \to B_1 \otimes \acute{B}_1$ be homomorphism topological module, $\acute{g} : A_1 \otimes \acute{A}_1 \to B_1 \otimes \acute{B}_1$ is the surjective topological module but $E_1 \otimes \acute{E}_1$ is topological module, i,e., $\exists\, f^{*'} : E_1 \otimes \acute{E}_1 \to A_1 \otimes \acute{A}_1$ is homomorphism topological module st: $\acute{g} \circ f^{*'} = \acute{f} \circ h$.
Now we define,

$$\acute{h} : E_3 \otimes \acute{E}_3 \to A_1 \otimes \acute{A}_1 (s.t) \acute{h} = f^{*'} \circ h^{-1},$$

we need to prove that,

$$\acute{g}\circ\acute{h}=\acute{f}\acute{g}\circ\acute{h}=\acute{g}\circ(f^{*'}\circ h^{-1})=(\acute{g}\circ f^{*'})\circ h^{-1},$$
$$=(\acute{f}\circ h)\circ h^{-1}(f\circ h)\circ h^{-1}=\acute{f}.$$

Hence, $E_3\otimes\acute{E}_3$ is topological FR–module (Figure 2).

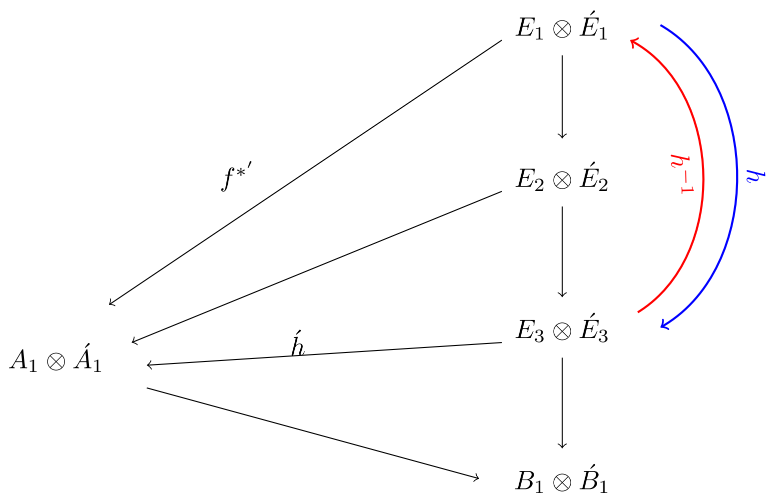

Figure 2: $E_3\otimes\acute{E}_3$ is topological FR- module.

□

Proposition 3.4. *If $E_i\otimes E_j$, $\otimes_{1\le i,j\le n}(E_i\otimes E_j)$ is left topological module of R st: $E_i\otimes E_j$ is topological FR–modules and $h : E_i\otimes E_j \to \otimes_{1\le i,j\le n}(E_i\otimes E_j)$ is homomorphism topological module, then $\otimes_{1\le i,j\le n}(E_i\otimes E_j)$ is topological FR–modules.*

Proof. By the diagram $A_i\otimes A_j$ and $B_i\otimes B_j$ are topological module of R and,

$$\acute{f} : E_i\otimes E_j\to B_i\otimes B_j,$$

homomorphism topological module s.t:

$$\acute{f}\circ h=\acute{f}\ ,\acute{g} : A_i\otimes A_j\to B_i\otimes B_j,$$

is surjective topological module but $E_i\otimes E_j$ is topological module, so $\exists\ f^{*'} : E_i\otimes E_j\to A_i\otimes A_j$ is homomorphism topological module s.t:

$$\acute{g}\circ f^{*'}=\acute{f}\circ h.$$

Now, we define $\acute{h} : \otimes_{1\le i,j\le n}(E_i\otimes E_j)\ \to A_i\otimes A_j$ s.t $\acute{h}=f^{*'}\circ h^{-1}$, we can prove

$$\acute{g}\circ\acute{h}=\acute{f}\acute{g}\circ\acute{h}=\acute{g}\circ\left(f^{*'}\circ h^{-1}\right),$$
$$=\left(\acute{g}\circ f^{*'}\right)\circ h=\acute{f}\circ\left(h\circ h^{-1}\right)=\acute{f}.$$

Thus, $\otimes_{1\le i,j\le n}(E_i\otimes E_j)$ is topological FR–modules. □

Proposition 3.5. *If,* $\otimes_{1\le i,j\le n}\left(E_i\otimes E_j\right), \otimes_{1\le p,q\le n}\left(E_p\otimes E_q\right)$ *are left topological modules of* R *s.t:* $\otimes_{1\le i,j\le n}E_i\otimes E_j$ *is topological* FR*–modules and* $h:\otimes_{1\le i,j\le n}E_i\otimes E_j\to\otimes_{1\le P,q\le n}(E_p\otimes E_q)$ *is homeomorphism topological module then,* $\otimes_{1\le P,q\le n}(E_p\otimes E_q)$ *is topological* FR*–modules.*

Proof. By the diagram, we obtain $\otimes\left(A\otimes\acute{A}\right)$ and $\otimes\left(B\otimes\acute{B}\right)$ two topological module and $\acute{f}:\otimes_{1\le i,j\le n}(E_i\otimes E_j)\to\otimes_{1\le P,q\le n}(E_p\otimes E_q)$ be homeomorphism topological module, $\acute{g}:A\otimes\acute{A}\to B\otimes\acute{B}$ is surjective topological module but,

$$f^{*'}:\otimes_{1\le i,j\le n}(E_i\otimes E_j)\to\otimes\left(A_1\otimes\acute{A}_1\right),$$

homomorphism topological module s.t $\acute{g}\circ f^{*'}=\acute{f}\circ h$. Now we define,

$$\acute{h}:\otimes_{1\le P,q\le n}(E_p\otimes E_q)\to\otimes\left(A\otimes\acute{A}\right),$$

such that $\acute{h}=f^{*'}\circ h$. We can prove,

$$\begin{aligned}\acute{g}\circ\acute{h}&=\acute{f}\acute{g}\circ\acute{h}=\acute{g}\circ\left(f^{*'}\circ h^{-1}\right),\\&=\left(\acute{g}\circ f^{*'}\right)\circ h^{-1}\left(\acute{f}\circ h\right)\circ h^{-1}\Rightarrow\acute{f}\circ\left(h\circ h^{-1}\right)=\acute{f}.\end{aligned}$$

Thus, $\otimes_{1\le P,q\le n}(E_p\otimes E_q)$ is topological FR–modules. □

Proposition 3.6. *If* $\otimes_{1\le\gamma_1\le n}E_{\gamma_1}$ *and* $\otimes_{1\le\gamma_2\le n}E_{\gamma_2}, \dots, \otimes_{1\le\gamma_n\le n}E_{\gamma_n}$ *are left topological modules, st* $\otimes E_{\gamma_1}, \dots, \otimes_{1\le\gamma_{n-1}\le n}E_{\gamma_{n-1}}$ *are* FR*-topological modules, and* $h:\otimes_{1\le\gamma_1\le n}E_{\gamma_1}\to\otimes_{1\le\gamma_n\le n}E_{\gamma_n}$ *be a homeomorphism of topological modules, then,* $\otimes_{1\le\gamma_n\le n}E_{\gamma_n}$ *are also topological* FR*–modules.*

Proof. By the diagram we obtain $A\otimes\acute{A}, B\otimes\acute{B}$ are topological module of R and

$$\acute{f}:\otimes_{1\le\gamma_1\le n}E_{\gamma_1}\to\otimes_{1\le\gamma_2\le n}E_{\gamma_2},$$

be homomorphism topological module and $\acute{g}:A\otimes\acute{A}\to B\otimes\acute{B}$ is a surjective topological module. But if $\otimes_{1\le\gamma_1\le n}E_{\gamma_1}$ is topological module, then there exists,

$$f^{*'}:\otimes_{1\le\gamma_1\le n}E_{\gamma_1}\to A\otimes\acute{A},$$

is the homomorphism topological module st $\acute{g}\circ f^{*'}=\acute{f}\circ h$. Now, we define,

$$\acute{h}:\otimes_{1\le\gamma_n\le n}E_{\gamma_n}\to A\otimes\acute{A},$$

where $\acute{h}=f^{*'}\circ h^{-1}$, also we could prove:

$$\begin{aligned}\acute{g}\circ\acute{h}&=\acute{f}\acute{g}\circ\acute{h}=\acute{g}\circ\left(f^{*'}\circ h^{-1}\right),\\&=\left(\acute{f}\circ h\right)\circ h^{-1}=\acute{f}\circ\left(h\circ h^{-1}\right)=\acute{f}.\end{aligned}$$

Thus, $\otimes_{1\le\gamma_n\le n}E_{\gamma_n}$ is topological FR–module. □

Proposition 3.7. *Topological* FR*–modules* $\otimes_{1\le i,j\le n}(E_i\otimes E_j)$ *of* R *is completely disconnected.*

Proof. Since period topological submodule $(E_i \otimes E_j)$ of the base of the topological module FR-modules $E \otimes \acute{E}$ is open of $E \otimes \acute{E}$, then all submodule contain component $E \otimes \acute{E}$ at zero of $E \otimes \acute{E}$ is totally disconnected. □

Proposition 3.8. *The topological FR–modules $\otimes_{1 \leq i,j \leq n} \left\{E_i \otimes E_j\right\}$ of R is totally disconnected.*

Proof. By Proposition 3.6. Let, $Z_1 \otimes Z_2$ and $W_1 \otimes W_2$ be two topological modules of R and let $\acute{f} : \otimes_{1 \leq \gamma_1 \leq n_1} \left(V_{\gamma_1} \otimes \acute{V}_{\gamma_1}\right) \to W_1 \otimes W_2$ be a topological homeomorphism module and

$$g : Z_1 \otimes Z_2 \to W_1 \otimes W_2,$$

is a topological surjective homeomorphism module. Since $\otimes_{1 \leq \gamma_1 \leq n_1} \left(V_{\gamma_1} \otimes \acute{V}_{\gamma_1}\right)$ is dual injective of topological module, then,

$$\exists \ f^{*'} : \ \otimes_{1 \leq \gamma_1 \leq n_1} V_1 \otimes \acute{V}_1 \to Z_1 \otimes Z_2,$$

is homeomorphism topological module. Define $\acute{h} : \otimes_{1 \leq \gamma_n \leq n_n} \left(V_{\gamma_n} \otimes \acute{V}_{\gamma_n}\right) \to Z_1 \otimes Z_2$ by,

$$\acute{h} \ (v \otimes \acute{v}) = f^{*'} \circ h \,(v \otimes \acute{v}) \ \forall \,(v \otimes \acute{v}) \in \otimes_{1 \leq \gamma_n \leq n_n} \left(V_{\gamma_n} \otimes \acute{V}_{\gamma_n}\right),$$

where $(v \otimes \acute{v}) = (v_1 \otimes \acute{v}_1) \ \otimes (v_2 \otimes \acute{v}_2), \ldots, \otimes (v_{\gamma_n} \otimes \acute{v}_{\gamma_n})$ and

$$\acute{g} \circ \acute{h}\,(v \otimes \acute{v}) = \acute{g} \circ f^{*'} \circ h\,(v \otimes \acute{v}) = \acute{f} \circ h\,(v \otimes \acute{v}) = \acute{f}\,(v \otimes \acute{v}).$$

Thus, $\otimes_{1 \leq \gamma_n \leq n_n} \left(V_{\gamma_n} \otimes \acute{V}_{\gamma_n}\right)$ is dual injective of topological module. □

4 Conclusion

This paper introduced a definition of topological free module and gone over many key results within the area of general topology. A basic introduction to topological groups, topological rings and topological modules is provided. In addition, we have given a review of the definition of topological module base and topological free module. The tensor concept is used to explain the new results. For future work, the definitions of topological free module and topological dual injuctive module could introduced by using the concept of topological base rather than general topology.

Acknowledgement Gratitude is extended to colleagues, institutions, and reviewers for their support, insights, and feedback, which contributed to the completion of this work.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] H. A. A. H. Abbas (2010). Topological direct projective modules. *Journal of Kerbala University*, *8*(1), 112–118.

[2] D. Dikranjan & M. Tkačenko (2001). Sequentially complete groups: dimension and minimality. *Journal of Pure and Applied Algebra*, *157*(2-3), 215–239. https://doi.org/10.1016/S0022-4049(00)00015-3.

[3] J. Gildea & S. O' Malley (2021). Units of the group algebra of the group $C_n \times D_8$ over any finite field of characteristic 2. *Malaysian Journal of Mathematical Sciences*, *15*(2), 303–312.

[4] P. J. Higgins (1975). *An Introduction to Topological Groups*. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9781107359918.

[5] H. Ibrahim & A. Khalaf (2021). Topological rings and modules via operations. *Iranian Journal of Mathematical Sciences and Informatics*, *16*(2), 31–48. http://dx.doi.org/10.52547/ijmsi.16.2.31.

[6] M. Mahuoub (2002). *On topological modules*. PhD thesis, Mustanseriyah University, Iraq.

[7] T. H. Majeed & M. J. Hussein (2013). On some result of topological projective modules. *Academic Science Journal*, *9*(3), 129–142.

[8] T. Nagaiah & M. R. Valluri (2022). A special class of rings. *Malaysian Journal of Mathematical Sciences*, *16*(1), 141–151. https://doi.org/10.47836/mjms.16.1.11.

[9] R. G. Swan (1977). Topological examples of projective modules. *Transactions of the American Mathematical Society*, *230*, 201–234. https://doi.org/10.2307/1997717.